MATH 2401 (2008)

a) Solve $f_x = f_y = f_z = 0$ & add constraint to find λ, $f = z - \lambda(2x^2 + 3y^2 + z^2 - 12xy + 4xz)$

$$\left.\begin{array}{lll} f_x = 0 & \Rightarrow & 4x - 12y + 4z = 0 \\ f_y = 0 & \Rightarrow & 6y - 12x = 0 \quad \Rightarrow y = 2x \end{array}\right\} \Rightarrow z = 5x \quad \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \Rightarrow \begin{array}{l} x = 1/14\lambda \\ y = 2/14\lambda \\ z = 5/14\lambda \end{array}$$

$$f_z = 0 \quad \Rightarrow \quad 1 - \lambda(2z + 4x) = 0$$

Constraint $\quad 2x^2 + 3y^2 + z^2 - 12xy + 4xz = 35 \quad \Rightarrow \left(\frac{1}{14\lambda}\right)^2 \left[2 + 12 + 25 - 24 + 20\right] = 35$

$$\Rightarrow \left(\frac{1}{14\lambda}\right)^2 = \frac{35}{35} = 1 \qquad \Rightarrow \quad \underline{\underline{z = \pm 5}}$$

b) i) $r^2 = x_j x_j \Rightarrow 2r\,\partial r/\partial x_i = 2x_j\,\partial x_j/\partial x_i = 2x_j \delta_{ji} = 2x_i \Rightarrow \underline{\frac{\partial r}{\partial x_i} = x_i/r}$

ii) $(\underline{h}\cdot\nabla)g(r) = h_i \frac{\partial}{\partial x_i} g(r) = h_i g'(r) \frac{\partial r}{\partial x_i} = h_i x_i g'(r)/r = \underline{h}\cdot\underline{x}\, g'(r)/r$

iii) $(\underline{h}\cdot\nabla)^2 g(r) = h_i \frac{\partial}{\partial x_i} h_j x_j g'/r = h_i h_j \left(\underbrace{\frac{\partial x_j}{\partial x_i}}_{\delta_{ij}} g'/r + x_j \frac{x_i}{r} (g'/r)' \right)$

$$= h_i h_i\, g'/r + \tfrac{1}{r}(g'/r)'\, h_i x_i h_j x_j$$

$$= \underline{h}^2 g'/r + 1/r\,(g'/r)'(\underline{h}\cdot\underline{x})^2$$

Taylors Theorem is $\quad f(\underline{x} + \underline{h}) = f(\underline{x}) + (\underline{h}\cdot\nabla)f + \frac{1}{2}(\underline{h}\cdot\nabla)^2 f + \dots$

Here $f(\underline{x} + \underline{h}) = g(|\underline{x} + \underline{h}|) = g(\underbrace{|\underline{x}|}_{r}) + (\underline{h}\cdot\nabla)g(r) + \frac{1}{2}(\underline{h}\cdot\nabla)^2 g(r) + \dots$

$$\underline{= g(r) + \underline{h}\cdot\underline{x}\, g'/r + \tfrac{1}{2}\left(\underline{h}^2 g'/r + (\underline{h}\cdot\underline{x})^2 (g'/r)'/r\right) + \dots}$$

as required.

2) a) Consider $\frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] = \frac{\partial F}{\partial y}\cdot y' + \cancel{\frac{\partial F}{\partial y'}y''} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \cancel{y''\frac{\partial F}{\partial y'}}$

$= y'\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] = 0$ from Euler's Eqn

$\Rightarrow F - y'\frac{\partial F}{\partial y'}$ is constant

b) Consider $\int_0^1 y'^2 + 4y - \lambda y \, dx$. its easiest to use original E-L eqns

$(4-\lambda) - \frac{d}{dx}(2y') = 0 \Rightarrow y'' = \frac{1}{2}(4-\lambda)$, Integrating & using b.c.

$y = \frac{1}{2}(4-\lambda)\frac{1}{2}(x^2 - x)$. Put in constraint, $4 = \frac{(4-\lambda)}{4}\int_0^1 x^2 - x\,dx$

$= \frac{4-\lambda}{4}\left(\frac{-1}{6}\right) \Rightarrow \frac{4-\lambda}{4} = -24$ & $y = 24(x - x^2)$. Hence

$I = \int_0^1 (24)^2(1-2x)^2 + 4.24(x - x^2)\,dx = (24)^2\left[1 - 2 + \frac{4}{3}\right] + 4.24.\left[\frac{1}{2} - \frac{1}{3}\right]$

$= 24.24/3 + 4.24/6 = 24(8 + 2/3) = 24.\frac{26}{3} = 8.26 = \underline{208}$

Students may be tempted to use Beltrami's equation.

$(y'^2 + 4y - \lambda y) - y'2y' = C \Rightarrow y'^2 = (4-\lambda)y - c$

$\Rightarrow \int \frac{dy}{\sqrt{(4-\lambda)y - c}} = \pm\int dx \Rightarrow \frac{2\sqrt{(4-\lambda)y - c}}{(4-\lambda)} = \pm(x + A) \Rightarrow (4-\lambda)y = c + \frac{(x+A)^2(4-\lambda)^2}{4}$

$y(0) = y(1) = 0 \Rightarrow A^2 = (1+A)^2,\ A = -\frac{1}{2}$ & $c = \frac{(4-\lambda)^2}{4}\cdot\left(\frac{-1}{4}\right)$

$y = \frac{(4-\lambda)}{4}\left[(x - \frac{1}{2})^2 - \frac{1}{4}\right]$

Q3.

a) $u_y + uu_x + \lambda u = 0 \qquad u(x,0) = f(x)$

x satisfy $\frac{dy}{dt} = 1, \quad \frac{dx}{dt} = u, \quad \frac{du}{dt} = -\lambda u, \quad x = s,\ y = 0,\ u = f(s)$ on $t = 0$

$\Rightarrow y = t + y_0$ & bc $\Rightarrow y = t, \quad u = u_0 e^{-\lambda t}$ & bc $\Rightarrow u_0 = f(s), \quad \frac{dx}{dt} = f(s)e^{-\lambda t}$

$\Rightarrow x = x_0 - \frac{f(s)}{\lambda}e^{-\lambda t}$ & bc $\Rightarrow x_0 = s + f(s)/\lambda$

$\Rightarrow y = t, \quad u = f(s)e^{-\lambda t}, \quad x = s + f(s)(1 - e^{-\lambda t})/\lambda$

Eliminate $t = y$ & s & use $f(s) = e^{\lambda t}u \Rightarrow u = e^{-\lambda y} f\left(x - ue^{\lambda y}\frac{(1 - e^{-\lambda y})}{\lambda}\right)$

b) $x(x+y)u_x + y(x+y)u_y = -(x-y)(2x+2y+u)$

x traces $\frac{dx}{dy} = \frac{x(x+y)}{y(x+y)} = \frac{x}{y} \Rightarrow y = \varphi x, \quad \varphi = y/x = \text{constant}$

$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x} + \frac{\partial}{\partial \varphi}(-\varphi/x), \quad \frac{\partial}{\partial y} \to \frac{\partial}{\partial \varphi}\cdot 1/x$

$\Rightarrow x(x + \varphi x)\left(u_x - \cancel{\frac{\varphi}{x}u_\varphi}\right) + \cancel{\varphi x(1 + \varphi x)\frac{1}{x}u_\varphi} = -(x - \varphi x)(2x + 2\varphi x + u)$

$\Rightarrow x^2(1+\varphi)u_x = -x^2(1-\varphi)\left(2(1+\varphi) + \frac{u}{x}\right)$

$\Rightarrow \frac{\partial u}{\partial x} + \frac{(1-\varphi)}{(1+\varphi)}\frac{u}{x} = -2(1-\varphi)$

IF is $x^{\frac{1-\varphi}{1+\varphi}} \Rightarrow \frac{d}{dx}\left[u x^{\frac{1-\varphi}{1+\varphi}}\right] = -2(1-\varphi)x^{\frac{1-\varphi}{1+\varphi}}$

$\Rightarrow u x^{\frac{1-\varphi}{1+\varphi}} = x^{\frac{1-\varphi}{1+\varphi}+1}(\varphi - 1)(\varphi + 1) + C(\varphi)$

$\Rightarrow u = x\left(y^2/x^2 - 1\right) + x^{\frac{y-x}{y+x}}C(y/x)$

4) $c^2 z_{xx} = z_{tt}$, Try $z = f(x+mt)$ & substitution yeilds

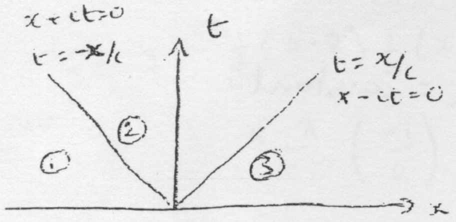
$c^2 f'' = m^2 f'' \Rightarrow m = \pm c$ & $z = f(x+ct) + g(x-ct)$

Imposing the initial conditions requires $f+g = F$ & $c(f'-g') = G \Rightarrow$

$$f-g = \frac{1}{c}\int_a^x G(\xi)\,d\xi \Rightarrow f = \frac{1}{2}F + \frac{1}{2c}\int_a^x G(\xi)\,d\xi, \quad g = \frac{1}{2}F - \frac{1}{2c}\int_a^x G(\xi)\,d\xi$$

$$\& \; f(x+ct) + g(x-ct) = \frac{1}{2}\{F(x+ct) + F(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$

If $F=0$ & $G(\xi) = \begin{cases} 1/1+\xi \\ 0 \end{cases}$, $z(x,t) = \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} \begin{matrix} 1/1+\xi & \xi > 0 \\ 0 & \xi < 0 \end{matrix} \; d\xi$



In ① $x-ct$ & $x+ct < 0$ & $z = 0$.

in ② $x-ct<0$, $x+ct>0$ & $z = \dfrac{1}{2c}\displaystyle\int_0^{x+ct} \frac{1}{1+\xi}\,d\xi$

$$= \frac{1}{2c}\Big[\ln(1+\xi)\Big]_0^{x+ct}$$

$$= \frac{1}{2c}\ln(1+x+ct)$$

in ③ $x+ct$ & $x-ct > 0$ & $z = \dfrac{1}{2c}\Big[\ln(1+\xi)\Big]_{x-ct}^{x+ct} = \dfrac{1}{2c}\ln\left(\dfrac{1+x+ct}{1+x-ct}\right)$.

Continuity needs to be checked ~~at~~ between ① & ②, $x+ct=0$ & $\ln 1 = 0$ ✓ $z=0$

② & ③, $x-ct=0$ & ~~to~~ $z = \frac{1}{2c}\ln(1+x+ct)$ ✓

The singularities in the logarithm are not attained

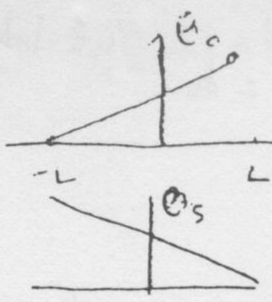
$\left.\dfrac{\partial z}{\partial t}\right|_{t=0} = 0$ in ① ✓ & $= \dfrac{1}{2c}\left\{c\dfrac{1}{1+x} - (-c)\dfrac{1}{1+x}\right\} = \dfrac{1}{1+x}$ in ③ ✓

As $x \to -\infty$ $z = 0$ identically. As $x \to +\infty$ $z \sim \dfrac{1}{2c}\left\{\ln\left(1+\frac{1+ct}{x}\right) - \ln\left(1+\frac{1-ct}{x}\right)\right\}$

$$\sim \frac{1}{2c}\cdot\frac{2ct}{x} \sim t/x \to 0$$

Q5.

$\alpha^2\theta_{xx} = \theta_t$ Steady solution is $\theta_0 = \frac{T}{2} + \frac{xT}{2L} = \frac{T}{2}(1 + x/L)$ (satisfies θ_{xx} & b.c.'s)



Write $\theta = \theta_s + \theta_u(x,t)$ then $\theta_s = \frac{T}{2}(1 - x/L)$ (5)

& $\theta_{u\,xx} = \theta_{ut}/\alpha^2$ & $\theta_u(L) = \theta_u(-L) = 0$, $\theta_u(0,t) = \theta_0 - \theta_s$
$= Tx/L$

If $\theta_u = \Theta(x)T(t)$ then $\Theta''/\Theta = T'/T\alpha^2 = -p^2$ say, chosen so oscillatory solutions occur in x, $\Theta(x) = A\cos px + B\sin px$. We will only need the $\sin px$ solutions as initial & boundary conditions are odd. The bc $\theta(L,t) = 0 \Rightarrow \Theta(L) = 0 \Rightarrow \sin pL = 0$
$\Rightarrow p = n\pi/L$. Solving for T & summing over possible solutions

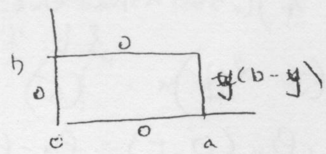
$$\theta(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-\alpha^2 n^2\pi^2 t/L^2} + \theta_s(x)$$

B_n chosen so that $\theta_0 - \theta_s = \frac{Tx}{L} = \sum_1^\infty B_n \sin\frac{n\pi x}{L} \Rightarrow B_n \frac{1}{2}.2L = \int_{-L}^{L} \frac{Tx}{L}\sin\frac{n\pi x}{L}$

$$\Rightarrow B_n = \frac{2T}{L^2}\int_0^L x\sin\frac{n\pi x}{L}\,dx = \frac{2T}{L^2}\left\{\left[-x\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right]_0^L + \frac{L}{n\pi}\int_0^L \cancel{\cos\frac{n\pi x}{L}}\,dx\right\} = -\frac{2T(-1)^n}{n\pi}$$

& $$\theta = \frac{T}{2}\left(1 - \frac{x}{L}\right) - 2T\sum_1^\infty e^{-\alpha^2 n^2\pi^2 t/L^2}\,.\,(-1)^n\frac{\sin n\pi x/L}{n\pi}$$

6)



Choosing oscillatory solutions in y & hyperbolic in x & taking into account the bc $u=0$ on $x=0$, $u=0$ on $y=b$

$$u=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{b}\sinh\frac{n\pi x}{b}$$

& the A_n are chosen so that $\displaystyle y(b-y)=\sum_{1}^{\infty} A_n \sinh\frac{n\pi a}{b}\sin\frac{n\pi y}{b}$, so that

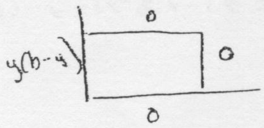
$$A_n\sinh\frac{n\pi a}{b}\cdot\frac{b}{2}=\int_0^b y(b-y)\sin\frac{n\pi y}{b}\,dy=b^3\int_0^1 s(1-s)\sin\alpha s\,ds,\quad \alpha=n\pi$$

$$=b^3\left\{\left[\cancel{s(1-s)\frac{\cos\alpha s}{-\alpha}}\right]_0^1+\frac{1}{\alpha}\int_0^1(1-2s)\cos\alpha s\,ds\right\}=\frac{b^3}{\alpha}\left\{\left[\cancel{(1-2s)\frac{\sin\alpha s}{\alpha}}\right]_0^1+\frac{2}{\alpha}\int_0^1\sin\alpha s\,ds\right\}$$

$$=\frac{2b^3}{\alpha^3}\left[-\cos\alpha s\right]_0^1=\frac{2b^3}{\alpha^3}\left[1-(-1)^n\right]\quad\&\quad A_n=\frac{4b^2}{n^3\pi^3}\frac{1}{\sinh\frac{n\pi a}{b}}$$

$$\&\quad u=\frac{4b^2}{\pi^3}\sum_{1}^{\infty}\frac{1-(-1)^n}{n^3}\sin\frac{n\pi y}{b}\cdot\frac{\sinh n\pi x/b}{\sinh n\pi a/b}$$

The solution to the problem ~~has~~ with $u=0$ on $x=a$, $u=y(b-y)$ on $x=0$ is as above but with $x\to a-x$ & so has same value on $x=a/2$.



The solution to the required problem is the sum of these two solutions. On $x=a/2$ & using $\dfrac{\sinh n\pi a/2b}{\sinh n\pi a/b}=\dfrac{1}{2\cosh n\pi a/2}$

$$u(a/2,y)=\frac{2.2\,b^2}{\pi^3}\sum_{1}^{\infty}\frac{1-(-1)^n}{n^3}\frac{\sin\frac{n\pi y}{b}}{\cosh n\pi a/2b}$$